United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2 : Thursday, 27 January 2011

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*
- To accommodate candidates sitting in other timezones, please do not discuss any aspect of the paper on the internet until 8am on Friday 28 January GMT.

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (14-18 April 2011). At the training session, students sit a pair of IMO-style papers and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's IMO (to be held in Amsterdam, The Netherlands 16–24 July) will then be chosen.

Do not turn over until **told to do so.**

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2010/11 British Mathematical Olympiad

Round 2

1. Let ABC be a triangle and X be a point inside the triangle. The lines AX, BX and CX meet the circle ABC again at P, Q and R respectively. Choose a point U on XP which is between X and P. Suppose that the lines through U which are parallel to AB and CA meet XQ and XR at points V and W respectively. Prove that the points R, W, V and Q lie on a circle.

2. Find all positive integers x and y such that $x + y + 1$ divides $2xy$ and $x + y - 1$ divides $x^2 + y^2 - 1$.

3. The function f is defined on the positive integers as follows;

$$\begin{aligned} f(1) &= 1; \\ f(2n) &= f(n) \text{ if } n \text{ is even;} \\ f(2n) &= 2f(n) \text{ if } n \text{ is odd;} \\ f(2n+1) &= 2f(n) + 1 \text{ if } n \text{ is even;} \\ f(2n+1) &= f(n) \text{ if } n \text{ is odd.} \end{aligned}$$

   Find the number of positive integers n which are less than 2011 and have the property that $f(n) = f(2011)$.

4. Let G be the set of points (x, y) in the plane such that x and y are integers in the range $1 \leq x, y \leq 2011$. A subset S of G is said to be *parallelogram-free* if there is no proper parallelogram with all its vertices in S. Determine the largest possible size of a parallelogram-free subset of G. *Note that a proper parallelogram is one where its vertices do not all lie on the same line*